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Effective method for construction of low-dimensional models for heat transfer process

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Abstract

A low-dimensional model based on the method of proper orthogonal decomposition (POD) and the method of polyargumental systems (MPS) for thermal conductivity problems with strongly localized source of heat has been presented. The key aspect of these methods is that they enable to avoid weak point of other projection methods, which consists in a priori choice of basis functions. It enables us to use the MPS method and the POD method as convenient means to construct low-dimensional models of heat and mass transfer problems.

1. Introduction

In spite of the imposing progress in computer engineering, the direct numerical simulation based on finite element methods or finite difference methods, is unable to ensure to the full extent the analyses of all aspects of behavior of the simulated phenomenon. It is more convenient (in point of fact, it is the only possibility) to carry out the analyses of model instability, finding bifurcation points, analyses of control parameters impact etc. resting upon low-dimensional (low-parameter) model. In fact, the basic classical results of hydrodynamics and heat transfer have been obtained on the basis of application of integral and projection methods to elementary power approximation of thermalphysic fields being studied.

One of the ways to effectively construct a low-dimensional model is the use of projection methods. When using a projection approach the study of a physical phenomenon comes to construction of the dynamic or algebraic model obtained while projecting solution on a priori selected basis. ''Practical analyticity'' of the solution obtained in this case enables us to perform the analysis of the model in explicit form. Inverse problems and control problems, which are important in applied sense, are also more easily analyzed within low-dimensional models.

While using projection methods (Galerkin, Kantorovich et al.) one of the principal points that at the bottom determine the ''quality'' of obtained approximate solution is the selection of a projection basis. The selection of a basis virtually designates not only the necessary number of terms in approximation representation but also the computational process stability. It is known that the choice of spline functions, as basis functions in the finite element method, made it possible to overcome an ill-conditioned problem for Ritza method [1]. For conjugate operators the following can be stated: the more ''energy affinity'' have the approximated operator and the operator whose eigenfunctions are basis functions the more effective is a projection method [2].

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Nomenclature

a,R,F	basis functions
Bi	Bio number, $\alpha \cdot r_H/\lambda$
f	function for approximation
M_i	eigenvalue
P	dimensionless productivity of source of heat, $q_v/q_{v\max}$
q_v	heat productivity of source of heat
$q_{v\max}$	maximum heat productivity of source of heat
r	radial coordinate
r_H	outer radius
r_B	inner radius
t	temperature
T	dimensionless temperature, $\lambda t/(q_{v\max} r_H^2)$
$\vec{x},\vec{s}$	vectors of coordinates

Greek symbols

α	heat transfer coefficient
φ	angular coordinate
φ^*	dimensionless angular coordinate, $\varphi^* = \varphi/\pi$
η	integral error
λ	thermal-conductivity coefficient
ρ	dimensionless radial coordinate, r/r_H
ρ_B	dimensionless inner coordinate, r_B/r_H
ρ^*	redundant dimensionless radial coordinate, $\rho^* = (\rho-\rho_B)/(1-\rho_B)$
τ	time
ψ	basis functions

Subscripts

i, j, k	mode numbers
N	number of modes

Superscripts

$''$	second derivation
$'$	first derivation

2. Method

The effective approach to construction of basis functions for approximation problem is a procedure proposed by Lumley [3] and denominated as the method of proper orthogonal decomposition (POD) or Karhunen–Loeve procedure. Originally, this approach was proposed as an attempt to show up coherent structures in a studied physical field. Eventually, Karhunen–Loeve procedure consists in finding of spatial basis by means of determination of eigenfunctions of a certain integral equation. This integral equation (governing equation of the method) is constructed by the approximation of correlation function determined with respect to approximated field. The obtained basis (empirical basis—according to POD method terminology) is orthogonal, and temporal coefficients computed on its basis are uncorrelated in time. This allows one to discuss the terms of such approximation as noninteracting on average modes, but this does not imply that the modes do not participate in short time interaction.

The application of Karhunen–Loeve procedure to the results of numerical simulations of hydrodynamics problems, including turbulence, proved its efficiency [4–8]. The number of terms (number of model's parameters) in approximation representation diminished by an order.

Virtually, POD method in the proper sense is not a simulation method for the processes described by differential equations. Nevertheless, it can be treated as the first step in construction of a mathematical model within one or another projection procedure based on empiric basis functions. By approximation of the results of direct numerical simulation of a studied process with POD method at any single set of control parameters it is possible, by means of obtained empiric basis and on the base of Galerkin method, to construct a dynamic model of the process [9–13].

The issue of basis functions selection is principal in projection method also known as the method of the polyargumental systems (MPS) [14–16]. At the heart of this method there underlies the idea of substantial development of Kantorovich method on the basis of a principle consisting the in completeness of functional reflection of all available information on the studied object. Owing to this fact the use of MPS method eliminates typical of other projection methods element of a priori choice of basis functions.

For the approximation problem MPS approach consists in the expansion of approximated field with respect to basis that is determined proceeding from the specificity of the field itself.

Suppose the field of a certain physical variable—$f(\tau,\vec{x})$. On having chosen a standard form of representation:

$$f = \sum_{n=1}^{N} a_i(\tau)\psi_i(\vec{x}), \tag{1}$$

where $\{a_i(\tau)\}$ and $\{\psi_i(\vec{x})\}$—beforehand unknown systems of functions, and equated having corresponding weights integrals of both members of (1) we obtain the following integral expressions:

$$\int f \cdot a_j \mathrm{d}\tau = \sum_i \psi_i(\vec{x}) \cdot \int a_i a_j \mathrm{d}\tau, \quad j = 1 \ldots N \tag{2a}$$

$$\int f \cdot \psi_k \mathrm{d}\vec{x} = \sum_i a_i(\tau) \cdot \int \psi_i \psi_k \mathrm{d}\vec{x}, \quad k = 1 \ldots N \tag{2b}$$

It is evident that the relation (2) corresponds to the conditions of the best mean-square approximation at form of representation (1). Sequential iteration between the members of systems (2a) and (2b) enables to find the desired approximation.

Now we are modifying the obtained relations. Let us take advantage of the fact that for two-variable function, any bilinear N-form of type (1) can be changed to equivalent form via linear substitute of the function of each argument, but now on the basis of two orthogonal function systems. Having presumed that in form (1) $\{a_i(\tau)\}$ and $\{\psi_i(\vec{x})\}$ are orthogonal function systems, we obtain the following system:

$$a_i(\tau) \int \psi_i \cdot \psi_i \mathrm{d}x = \int f \cdot \psi_i(\vec{x}) \mathrm{d}\vec{x}, \quad i = 1 \ldots N \tag{3a}$$

$$\psi_k(\vec{x}) \int a_k \cdot a_k \mathrm{d}\tau = \int f \cdot \psi_k(\vec{x}) \mathrm{d}x, \quad k = 1 \ldots N \tag{3b}$$

Here, unlike system (2), each term in the form of representation (1) is found separately.

Let us revert to the POD method. The governing integral equation of this method can be written as follows:

$$M_i \cdot \psi_i(\vec{x}) = \iint (f(\tau, \vec{x}) f(\tau, \vec{s})) \mathrm{d}\tau \cdot \psi_i(\vec{s}) \mathrm{d}\vec{s} \tag{4}$$

If we substitute relation (3a) in (3b), as a result we will obtain an integral equation:

$$M_i \cdot \psi_i(\vec{x}) = \iint (f(\tau, \vec{x}) f(\tau, \vec{s})) \mathrm{d}\tau \cdot \psi_i(\vec{s}) \mathrm{d}\vec{s} \tag{5}$$

$$M_i = \int a_i a_i \mathrm{d}\tau \cdot \int \psi_i \psi_i \mathrm{d}\vec{x} \tag{6}$$

which, except for a single principal feature, is similar to the POD method integral equation. In the MPS method taking into account of conjunction (6) occurs in iteration process within system (3), therefore the value of M_i becomes a result of co-ordination of spatio-temporal behavior. For the POD method the value M_i is found as an eigenvalue of integral equation and is purely mathematical compatibility condition of the integral equation (4). This determines more transparency in implementation of the MPS method.

One can suggest further generalization of the MPS approach to approximation problems. As a criterion of the approximation representation optimality (that in this case study was a mean-square criterion) one can consider the criteria based on mean-square deviation of derivation or combined criteria including value of approximated field at boundaries, integral characteristics computed on the basis of approximated field etc. For the POD method a said generalization will consist in the modification of correlation function being a kernel of integral equation (4).

The MPS method, in respect of its initial statement, is a method of process simulation governed by differential equations. It is of interest to consider the simulation of a specific problem based on method of the polyargument systems, both from the point of view of approximation capability and from the point of view of the construction of low-dimensional (low-parameter) process model.

3. Results and conclusions

Let us consider a boundary problem of thermal conductivity for annular domain ($\rho_B \leqslant \rho \leqslant 1, 0 \leqslant \varphi \leqslant \pi$) described by Poisson's equation:

$$AT(\rho, \varphi) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial T}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 T}{\partial \varphi^2} = -P(\rho, \varphi), \tag{7}$$

at the following boundary conditions:

$$\begin{aligned} &\frac{\partial T}{\partial \rho} = 0 \quad \text{at } \rho = \rho_B; \\ &\frac{\partial T}{\partial \rho} + \mathrm{Bi} T = 0 \quad \text{at } \rho = 1, \\ &\frac{\partial T}{\partial \varphi} = 0 \quad \text{at } \varphi = 0, \ \varphi = \pi \end{aligned} \tag{8}$$

As a source of heat we consider

$$P(\rho, \varphi) = \{\sin \pi \rho^* \cdot \sin \pi(\varphi^* - 1/8)\}^{120}, \tag{9a}$$

$$\rho^* = (\rho - \rho_B)/(1 - \rho_B), \ \varphi^* = \varphi/\pi \tag{9b}$$

describing localized heat supply.

Having solved the stated problem by Fourier method, we obtain desired temperature field. The approximation of this field by expansion in Fourier series needs roughly several hundreds of terms to ensure an error of the order of a few per cent. At the same time the approximation built upon the basis of MPS method (or Karhunen–Loeve procedure) ensures such accuracy of the solution with four terms. It indicates the possibility to "catch" the process under study in low-parameter (low-term) form of representation. Let us notice that here in contrast to the first part of the paper we are considering the approximation with respect to two spatial coordinates.

Relying on the fact of adequacy of a few terms of the two-component form of representation to describe the problem:

$$T_N(\rho,\varphi)=\sum_i^N R_i(\rho)\cdot F_i(\varphi) \tag{10}$$

We apply MPS method procedure, now directly, to the differential equation (7) and (8). According to the principle of minimum apriority in the form of solution representation, both the systems of basis function $\{R_i\}$ are regarded and $\{F_i\}$ is unknown (that distinguishes this method from standard projection method). Further, conforming to the principle of functional reflection completeness of available information we project residual (after substitution of form (10) in (7) and (8)) on both the systems of basic functions.

Let us write the system of MPS method in operational notation:

$$\int_0^\pi \left(A\sum_i^N R_i(\rho)F_i(\varphi)-P(\rho,\varphi)\right)\cdot F_k(\varphi)\mathrm{d}\varphi=0$$
$$k=1\ldots N \tag{11a}$$

$$\int_{\rho_B}^1 \left(A\sum_i^N R_i(\rho)F_i(\varphi)-P(\rho,\varphi)\right)\rho\cdot R_j(\rho)\mathrm{d}\rho=0$$
$$j=1\ldots N \tag{11b}$$

In the same way as for system (2) we use iteration method to solve system (11). Unlike approximation system of the MPS method (2), system (11) cannot be reduced to the form solved with regard to each term in (10).

System (11) for discussed specific thermal conductivity problem in annular domain has the following form:

$$\sum_i^N\left\{\left[R_i''(\rho)+\frac{1}{\rho}R_i'(\rho)\right]I_{1ik}+\frac{1}{\rho^2}R_i(\rho)I_{2ik}\right\}=-I_{3k}(\rho),$$
$$k=1\ldots N$$

$$R_k'(\rho_B)=0,\quad R_k'(1)+\mathrm{Bi}\cdot R_k(1)=0,$$

$$\sum_i^N[F_i'(\varphi)J_{1ij}+(J_{2ij}+J_{3ij})\cdot F_i(\varphi)]=-J_{4j}(\varphi),$$
$$j=1\ldots N$$

$$F_j'(0)=F_j'(\pi)=0.$$

Here I_{1ik}, I_{2ik}, J_{1ij}, J_{2ij}, J_{3ij} and $I_{3k}(\rho)$, $J_{4j}(\varphi)$-reduced coefficients and right-hand members of one-dimensional equations:

$$I_{1ik}=\int_0^\pi F_i(\varphi)F_k(\varphi)\mathrm{d}\varphi,$$
$$J_{1ij}=\int_{\rho_B}^1\frac{1}{\rho}R_i(\rho)R_j(\rho)\mathrm{d}\rho,$$
$$I_{2ik}=\int_0^\pi F_i''(\varphi)F_k(\varphi)\mathrm{d}\varphi,$$
$$J_{2ij}=\int_{\rho_B}^1 R_i'(\rho)R_j(\rho)\rho\mathrm{d}\rho,$$
$$I_{3k}(\rho)=\int_0^\pi P(\rho,\varphi)F_k(\varphi)\mathrm{d}\varphi,$$
$$J_{3ij}=\int_{\rho_B}^1 R_i''(\rho)R_j(\rho)\rho\mathrm{d}\rho,$$
$$J_{4j}(\varphi)=\int_{\rho_B}^1 P(\rho,\varphi)R_j(\rho)\rho\mathrm{d}\rho$$

The problem was solved at $\rho_B=0.4$, Bi=10 and a heat source having the form (9a).

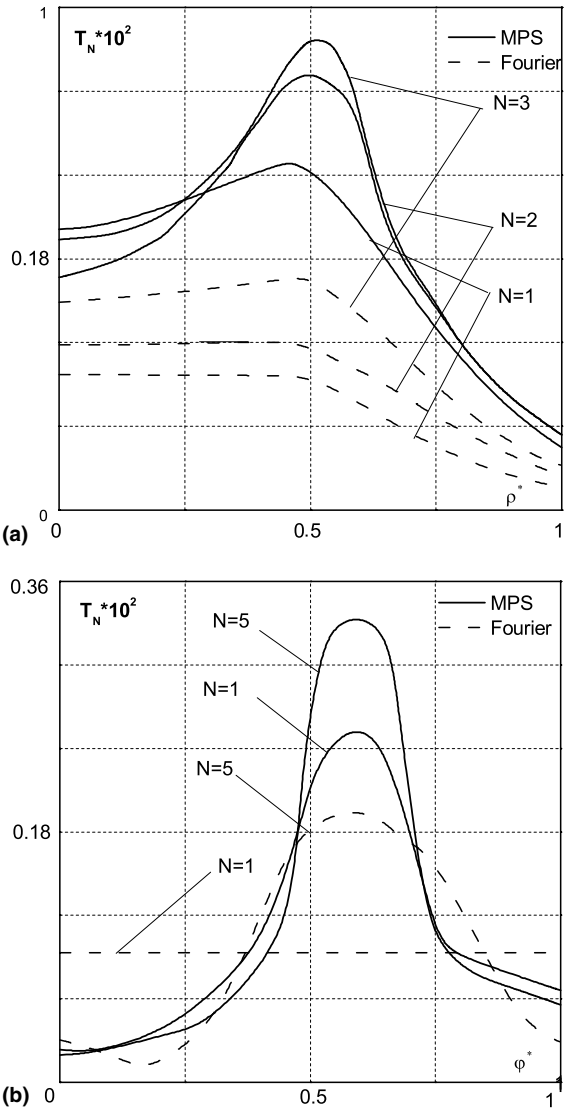

Fig. 1. (a) The temperature distribution $T_N(\rho^*,\varphi^*)$ at $\varphi^*=0.625$; ρ^*, φ^*- formula (9b). (b) The temperature distribution $T_N(\rho^*,\varphi^*)$ at $\rho^*=0.5$; ρ^*, φ^* - formula (9b).

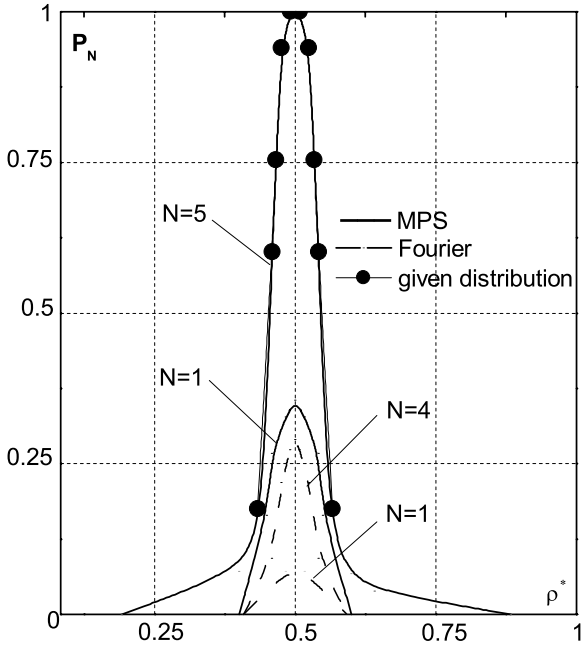

Fig. 2. The distribution $P_N(\rho^*,\varphi^*) = AT_N$ at $\varphi^* = 0.625$; ρ^*,φ^* -formula (9b).

In Fig. 1a and b there is shown a temperature distribution in the two sections of annular domain which pass through the point of maximal heat-source power. In Fig. 2 there is shown a plot of function $P_N(\rho,\varphi) = AT_N(\rho,\varphi)$ for various number of terms in solution.

Function $T_N(\rho,\varphi)$ found by MPS method ensures virtually exact solution with respect to T at $N = 3$, and for function P_N (ρ,φ) the coincidence with a given $P(\rho,\varphi)$ (formula (9a)) takes place already at $N = 5$. In Fig. 3 there is shown a dependence of integral error for the MPS method and Fourier method. In this figure the integral error are determined by the formula:

$$\eta = \int_0^{\pi}\int_{\rho_B}^{1} | P(\rho,\varphi) - AT_N(\rho,\varphi) || T_N | \cdot\rho\cdot \mathrm{d}\rho\cdot \mathrm{d}\varphi/\omega,$$
$$\omega = \int_0^{\pi}\int_{\rho_B}^{1} | P || T_N | \cdot\rho\cdot \mathrm{d}\rho\cdot \mathrm{d}\varphi \qquad (12)$$

In order to ensure error η at a level of 5–6% in expansion method with respect to two-dimensional eigenfunctions and in Fourier method (Galerkin method), 480 and 22 terms of a series are needed, but in MPS method one needs only 5 terms.

Shown results demonstrate that the analyzed thermal conductivity problem with strongly localized source of heat may be considered on the basis of a low-dimensional model described by the system of ordinary differential equations.

The selection of basis functions not a priori, but based on the features of the problem itself, aspiration for complete reflection of all the available information—these are principles underlying the MPS method and POD method. It enables the MPS method and the POD method, which is inherently related with the latter, serve as convenient means to construct low-dimensional models of heat and mass transfer problems.

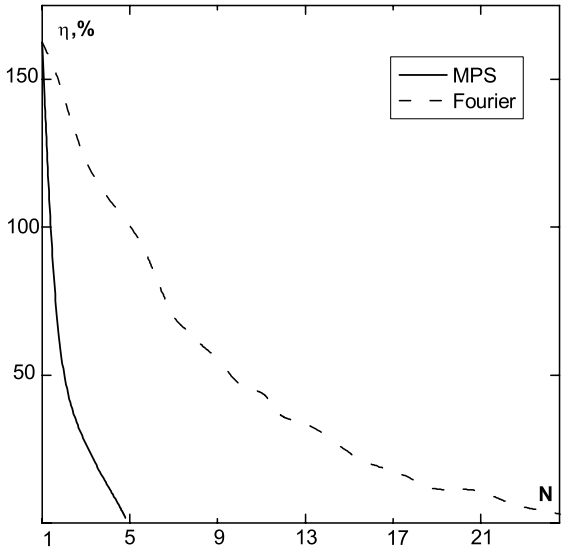

Fig. 3. The error η (formula (12)) versus the number of terms N.

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